NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

Rensselaer Polytechnic Institute

Troy, New York

STABILITY OF FLOW BETWEEN ARBITRARILY SPACED CONCENTRIC CYLINDRICAL SURFACES INCLUDING THE EFFECT OF A RADIAL TEMPERATURE GRADIENT

64-5

by

J. Walowit, S. Tsao, R. C. DiPrima

Contract Nonr-591(08)
Mechanics Branch
Office of Naval Research

at

Department of Mathematics
Rensselaer Polytechnic Institute
Troy, New York

August 14, 1963
RPI MathRep. No. 61

STABILITY OF FLOW BETWEEN ARBITRARILY SPACED CONCENTRIC CYLINDRICAL SURFACES INCLUDING THE EFFECT OF A RADIAL TEMPERATURE GRADIENT

by

J. Walowit, S. Tsao, R. C. DiPrima

ABSTRACT

The stability of Couette flow and flow due to an azimuthal pressure gradient between arbitrarily spaced concentric cylindrical surfaces is investigated. The stability problems are solved by using the Galerkin method in conjunction with a simple set of polynomial expansion functions. Results are given for a wide range of spacings. For Couette flow, in the case that the cylinders rotate in the same direction, a simple formula for predicting the critical speed is derived. The effect of a radial temperature gradient on the the stability of Couette flow is also considered. It is found that positive and negative temperature gradients are destabilizing and stabilizing respectively.

INTRODUCTION

The stability of viscous flow between concentric rotating cylinders was first investigated experimentally and theoretically by G. I. Taylor in his famous paper in 1923 [1].* He showed, in agreement with experiment, that when a certain criterion was violated a secondary motion in the form of toroidal vortices sets in. For the most part Taylor's computations were limited to the case when $\mu = \Omega_2/\Omega_1 > 0$ or only slightly negative (with a few exceptions) and $\eta = R_1/R_2 \sim 1$ where R_1, R_2 and Ω_1, Ω_2 are the radii and angular velocities of the inner and outer cylinders respectively. However, he did give a correction formula suitable for $1-\eta$ small. Since this original investigation, there have been a number of papers dealing with suitable mathematical techniques for solving the related eigenvalue problem for the "small-gap" problem ($\eta \to 1$), but for arbitrary rotation rates.** The case of $-\mu$ large is particularly difficult. (See [28], [2]). On the other hand, the "finite-gap" problem ($\eta \neq 1$) has only recently been considered. One of the difficulties in treating the finite-gap problem is that the differential operators in the eigenvalue problem has variable coefficients in contrast to the constant coefficient operators that appear in the small-gap problem.

Using an expansion technique similar to that for the small-gap problem in which one of two equations is solved exactly, Chandrasekhar [3] has considered the finite-gap problem. The computations are rather tedious and numerical

---

*Numbers in brackets designate References at the end of the paper.

**For a discussion of some of this work as well as references to other papers, the reader is referred to Chandrasekhar [2, Sec. 71].

results were obtained only for the case $\eta = \frac{1}{2}$, but for a wide range of values of μ, $-0.5 \leq \mu \leq 0.25$. (The flow is stable for $\mu > \eta^2$.) More recently Chandrasekhar [4] and Chandrasekhar and Elbert [5] have shown that some of the numerical difficulties can be overcome by considering the corresponding adjoint eigenvalue problem. Witting [6], treating $1-\eta$ as a small parameter, has also considered the stability problem by expanding in powers of $(1-\eta)$. Finally Kirchgässner [7] has constructed the Greens function for the finite-gap problem, and has solved the resultant integral equation by an iteration technique. Results were obtained for $\frac{1}{2} \leq \eta < 1$ with $\mu = 0$, $-0.4 \leq \mu \leq 0.25$ for $\eta = \frac{1}{2}$, and for $-0.4 \leq \mu \leq 4/9$ for $\eta = 2/3$.

A similar type of instability problem occurs when a viscous fluid flows in a curved channel under a pressure gradient acting round the channel. The small gap form of this stability problem was first considered by Dean [8]. The finite-gap Dean problem has not been considered in the literature. However, Meister [9] has considered the more difficult finite-gap combined Taylor-Dean problem for the cases $\eta = 3/4$ and $5/6$. By properly rescaling his results it is possible to obtain approximately the criterion for the Dean stability problem for these values of η. However, there is insufficient data to determine this limit accurately.

In the present paper the finite-gap Taylor and finite-gap Dean problems are considered. It is shown that each of these problems can be solved using the Galerkin method, the eigenfunctions being represented as a combination of simple polynomials. With this choice of expansion functions all necessary integrals can be easily evaluated and numerical computations have been carried out for a much wider range of η $(0.1 \leq \eta \leq 1)$ than in previous

computations. The results are in good agreement with the known theoretical results and all of the available experimental results.

Finally the effect of a radial temperature on the stability of Couette flow between rotating concentric cylinders is considered. The corresponding theoretical small-gap problem has been considered by Yih [10], Lai [11], and Becker and Kaye [12]. Experimental results for gases have been obtained by Becker and Kaye [13] and Bjorklund and Kaye [20], and for liquids by Nissan and Haas [29] and Nissan, Ho and Nardacci [30]. With just a few additional computations this problem can be solved by the Galerkin method used in solving the Taylor and Dean stability problems.

THE STABILITY PROBLEMS

Let r, θ, z denote the usual cylindrical coordinates, with the z axis coinciding with the axis of the cylinders; and let R_1, R_2 and Ω_1 and Ω_2 denote the radii and angular velocities of the inner and outer cylinders respectively. If u_r, u_θ, and u_z denote the components of velocity in the increasing r, θ, and z directions and p denotes the pressure, the Navier-Stokes equations admit a steady solution of the form

$$u_r = u_z = 0 \ , \qquad u_\theta = V(r) \ , \qquad \partial p/\partial r = \rho V^2/r \tag{1}$$

where ρ is the density.

Now superimpose* on this steady motion a small rotationally symmetric disturbance of a form such that the θ component of the velocity is

$$u_\theta(r,z,t) = V(r) + v(r)\, e^{\sigma t} \cos \lambda z \ . \tag{2}$$

*A complete derivation of the disturbance equations can be found in Chapter 7 of reference [2].

The motion will be unstable if there exist solutions of the resultant eigenvalue problem for which the real part of σ is greater than zero; it will be stable if all the solutions have real part of σ less than zero. It is known from experimental evidence that not only is the instability of this spatial form, but that also a new steady secondary motion occurs. Hence we can anticipate that the marginal state is given by $\sigma = 0$ rather than just real part $\sigma = 0$. In this case the linearised equations for the disturbance velocities are

$$(DD^* - \lambda^2)^2 u = \frac{2\lambda^2}{\nu} \Omega(r) v \tag{3}$$

$$(DD^* - \lambda^2) v = \frac{1}{\nu}(D^*V) u \tag{4}$$

where u is the disturbance velocity in the radial direction, ν is the kinematic viscosity, $\Omega(r) = V(r)/r$, and

$$D = \frac{d}{dr} , \qquad D^* = D + \frac{1}{r} \tag{5}$$

The requirement of no slip at the boundaries gives the boundary conditions

$$u = v = Du = 0 \tag{6}$$

at $r = R_1$ and $r = R_2$. We will consider two examples.

<u>The Taylor stability problem</u>. In this case the basic motion is that due to the motion of the cylindrical walls (Couette flow), and V(r) is given by

$$V(r) = Ar + \frac{B}{r} \tag{7}$$

where

$$A = \frac{\Omega_1 R_1^2 - \Omega_2 R_2^2}{R_1^2 - R_1^2} \quad , \quad B = R_1^2 R_2^2 \frac{\Omega_1 - \Omega_2}{R_2^2 - R_1^2} \; . \tag{8}$$

Notice that $D^*V = 2A$, a constant.

It is convenient to choose our length scale so that the range of the independent variable is independent of the parameters of the problem. Further, the natural scale, except in the limiting case $\mu = \Omega_2/\Omega_1 \rightarrow -\infty$ (See [28].), for λ is the gap width. Hence we introduce the dimensionless variables

$$r = R_o + dx \quad , \quad a = \lambda d \tag{9}$$

where $d = R_2 - R_1$ and $R_o = (R_2+R_1)/2$. Then the Taylor stability problem takes the form

$$(DD^* - a^2)^2 \, u = - a^2 \, T \, g(x) \, v \tag{10}$$

$$(DD^* - a^2) \, v = u \tag{11}$$

where

$$\Omega(r) = \Omega_1 \, g(x) \quad , \quad g(x) = A_1 + \frac{B_1}{\xi^2}$$

$$\xi = \frac{r}{R_2} = \eta + (1-\eta)(x+\tfrac{1}{2}) \quad , \quad \eta = R_1/R_2 \quad , \quad \mu = \Omega_2/\Omega_1 \tag{12}$$

$$A_1 = \frac{\mu - \eta^2}{1 - \eta^2} \quad , \quad B_1 = \eta^2 \frac{1 - \mu}{1 - \eta^2}$$

and T is the usual Taylor number

$$T = - 4A \, \Omega_1 d^4/\nu^2 \tag{13}$$

and we now use D and D^* for the dimensionless operators

$$D = \frac{d}{dx} \quad , \quad D^* = \frac{d}{dx} + \frac{1-\eta}{\xi} \ . \tag{14}$$

Finally u has been replaced by $(2Ad^2/\nu)^{-1}$ u.

Equations (10) and (11) with the boundary conditions

$$u = v = Du = 0 \tag{15}$$

at $x = \pm\frac{1}{2}$ determine an eigenvalue problem of the form

$$F(T, a, \mu, \eta) = 0 \ . \tag{16}$$

For fixed values of μ and η, which determine the geometry and basic velocity distribution up to scale factors, we wish to determine the minimum positive T over all real positive a for which there exists a non-trivial solution of the eigenvalue problem. The critical value of T determines the critical speed Ω_1 for fixed μ and η, and the corresponding critical value of a determines the spacing of the vortices in the z direction.

It should be noted that the Taylor number T as defined by Eq. (13) is different from that used by Chandrasekhar [3] ($T = -4ABR_2^2/\nu^2$) who used R_2 as his reference length. The present choice is the usual Taylor number for the small-gap problem; hence, in the small-gap limit $\eta \to 1$, Eqs. (10) and (11) reduce to the usual small-gap equations. Further we note that in the case $\mu = 0$, which is of most engineering interest, both A_1 and B_1 are proportional to η^2. Hence to obtain a meaningful problem for $\mu = 0$ as $\eta \to 0$ it is necessary to rescale T by a factor of η^2 treating $T\eta^2$ as the new eigenvalue. That is, we can anticipate that for $\mu = 0$, $T \to C/\eta^2$ where C is a constant as $\eta \to 0$. Most of our results are given in terms of $T\eta^2$ which also

reduces to the small-gap Taylor number as $\eta \to 1$.

The Dean stability problem. In this case the cylinder walls are at rest and the motion is due to a constant circumferential pressure gradient. The basic velocity is

$$V(r) = \frac{1}{2\rho\nu}\frac{\partial p}{\partial\theta}(r \ln r + Cr + \frac{E}{r}) \tag{17}$$

where

$$C = -\frac{R_2^2 \ln R_2 - R_1^2 \ln R_1}{R_2^2 - R_1^2} \quad , \quad E = \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \ln \frac{R_2}{R_1} \quad . \tag{18}$$

With x and a defined by Eqs. (9), and ζ, D and D* defined by Eqs. (12) and (14), the finite-gap Dean stability problem takes the form

$$(DD^* - a^2)^2\, u = a^2\, P\left[h(x)/\zeta\right]\, v \tag{19}$$

$$(DD^* - a^2)\, v = (D^* h)\, u \tag{20}$$

where

$$V(r) = V_m\, h(x) \quad , \quad V_m = -\frac{R_2}{2\rho\nu}\frac{\partial p}{\partial\theta}\frac{(1-\eta^2)^2 - 4\eta^2(\ln\eta)^2}{4(1-\eta)(1-\eta^2)}$$

$$h(x) = \frac{4(1-\eta)}{4\eta^2(\ln\eta)^2 - (1-\eta^2)^2}\, \zeta \left\{(1-\eta^2)\ln\zeta + \eta^2 \ln\eta\, (1 - \frac{1}{\zeta^2})\right\} \tag{21}$$

$$P = 2\left(\frac{V_m d}{\nu}\right)^2 \frac{d}{R_2} \quad .$$

and V_m is the mean velocity. Finally u has been replaced by $(V_m d/\nu)^{-1}$ u.

The boundary conditions are given by Eqs. (15).

In the small-gap limit, $\eta \to 1$, Eqs. (19) and (20) reduce to the usual small-gap equations within an integral multiplier of P.

METHOD OF SOLUTION

The eigenvalue problems for the finite-gap Taylor and Dean instability problems can be conveniently solved by the Galerkin method. The functions u and v are expanded in complete sets of functions satisfying the boundary conditions (15). The coefficients in the series are determined by the requirement that the error in Eqs. (10) and (11), for the Taylor problem, or in Eqs. (19) and (20), for the Dean problem, be orthogonal to the expansion functions for u and v respectively. This leads to a system of infinitely many linear, homogeneous equations for the coefficients in the series. For a nontrivial solution it is necessary that the determinant of the system of equations vanish, and this gives a determinantal equation for $T(a, \mu, \eta)$ or $P(a, \eta)$. In practice only a finite number of terms are used in the series for u and v, say M, and this leads to a determinant of size 2M.*

In principle there are a number of possible sets of complete functions which can be used for the expansion of u and v. For example, the set of functions $\sin(2n-1)\pi x$ and $\cos 2n\pi x$ $(n=1,2,\ldots)$ could be used for $v(x)$, and the set of functions $C_n(x)$ and $S_n(x)$ tabulated by Harris and Reid [32] could be used for $u(x)$. Unfortunately, the choice of trignometric functions leads to a prohibitive amount of labor just to evaluate the entries that appear in the determinantal equation. Indeed it is not clear that many of the integrals can

*It is not actually necessary that the number of terms in the series for u and v be the same.

even be evaluated in closed form.

For our purposes it is more convenient, and equally as satisfactory, to use simple polynomials in x that satisfy the boundary conditions. Thus we choose

$$u(x) = \sum_{n=1}^{\infty} \alpha_n u_n(x) \quad , \quad v(x) = \sum_{n=1}^{\infty} \beta_n v_n(x) \tag{22}$$

where $u_n(x) = (x^2 - \frac{1}{4})^2 x^{n-1}$ and $v_n(x) = (x^2 - \frac{1}{4}) x^{n-1}$. These polynomial expansion functions have also been used by Kurzweg [31] in his treatment of several small-gap problems in hydrodynamic and hydromagnetic stability. It is particularly interesting to note that with this choice of expansion functions all of the various integrals can be expressed exactly as polynomials in η or as the products of $\ln \eta$ and polynomials in η . In the use of the Galerkin method for these problems it is helpful to note that the operator $d/dr(d/dr + r^{-1})$ is a Sturm-Liouville type operator with a corresponding weight function r, or in our dimensionless variables ζ . Thus we require, for example in the Taylor problem, that the error in the equation for u be orthogonal to $\zeta u_m(x)$, $m=1,2,\ldots,M$, which gives

$$\sum_{n=1}^{M} \left\{ \alpha_n \left(\zeta u_m, (DD^* - a^2)^2 u_n \right) + \beta_n a^2 T \left(\zeta u_m, g(x) v_n \right) \right\} = 0 ,$$

$$m = 1,2,\ldots. \tag{23}$$

where (u_m,u_n) denotes the integral of $u_m u_n$ from $x = -\frac{1}{2}$ to $x = \frac{1}{2}$. Similarly for the equation for v(x) and for the Dean stability problem. The number of computations that are required are considerably reduced by the symmetries

$(\zeta u_m, (DD^*-a^2)^2 u_n) = (\zeta u_n, (DD^*-a^2) u_m)$, etc. and by the fact that the differential operators for the Taylor and Dean stability problems are the same.

As mentioned earlier, while it is possible to evaluate all integrals exactly, it is actually more convenient in practice to proceed in a slightly different manner. The details of the evaluation of the integrals are given in the Appendix. The determinant is evaluated by the method of pivotal condensation, the zero being found by trial and error. All of the computations were carried out on an I.B.M. 650.

For the Taylor stability problem computations were carried out for $M = 2$, 3, and 4 for a range of values of η from 1 to 0.1 and for a range of $\mu < \eta^2$. For $\mu > \eta^2$ the flow is known to be stable to disturbances of the form (2). The results are tabulated in Table 1. For $\eta = 1$, and μ in the range $-1 \leq \mu \leq 1$ the results for $M = 4$ are within 0.2% of the "exact results" given by Chandrasekhar [2, Sec. 71]. Indeed even with $M = 2$ the error for the worst case, $\mu = -1$, is only 5%. If we bear in mind that the analytical and numerical work using the Galerkin method with $M = 2$ is much less than that for the expansion procedure of Chandrasekhar or the integral equation technique, this result is surprisingly good. For $\eta = \frac{1}{2}$ and a range of μ, $-\frac{1}{2} \leq \mu \leq \frac{1}{4}$, and $M = 4$ the maximum difference between the present results and those of Chandrasekhar and Elbert [5] occurs at $\eta = -\frac{1}{2}$ and is about 2%. With decreasing η (wider gap) and decreasing μ it becomes more difficult to approximate the eigenfunction, so that at $\eta = 0.2$ it was not possible to determine T accurately for negative values of μ without taking more terms in the series for u and v. By comparing the results for $M = 2$, 3, and 4 it is estimated that all the results given in Table 1 for $M = 4$ are correct within 2%. For $\mu = 0$,

and a range of η, $0.2 \leq \eta \leq 1$, some recent results by Roberts [14] using direct numerical integration are essentially identical with those given here.

For the Dean stability problem computations have been carried out for a range of η, $0.1 \leq \eta \leq 1$. The results are tabulated in Table 2. The only known results for this problem are for $\eta = 1$. In this case the result for $(V_m d/\nu)(d/R_2)^{\frac{1}{2}}$ is essentially identical with the value 35.94 reported by Reid [15] and the values given by Hämmerlin [16], and Dean [8]. For M = 2 the difference is only 2.6%. Comparison of the results for M = 2 and 4 indicates that the error increases with decreasing η, the percentage change being about 7% at $\eta = 0.1$. It should be noted that the percentage change in the actual eigenvalue P appearing in Eq. (19) is twice that in $(V_m d/\nu)(d/R_2)^{\frac{1}{2}}$.

EFFECT OF A RADIAL TEMPERATURE GRADIENT

In this section we will consider the effect of a radial temperature gradient on the finite-gap Taylor stability problem. The presence of the radial temperature gradient will give rise to convective effects through the interaction of the associated radial density gradient with the radial acceleration. In this analysis we will neglect the effect of gravity. Since in both gases and liquids the centrifugal force field V^2/r corresponding to the critical speed is often much less than g, this is a serious approximation.* However, for stationary cylinders no significant changes in the heat transfer as predicted by conduction were observed in the measurements of Kraussold [17] for Rayleigh numbers (based on g) less than 10^3. For horizontal cylinders this was verified by the experiments of Liu, Mueller, and Landes [18] and the

*The ratio $(R_1 \Omega_1)^2/R_1 g$ is given approximately by $T(\nu^2/d^3)\eta(1+\eta)/4g(\eta^2-\mu)$.

theoretical analysis of Crawford and Lemlich [19]. In addition, in the experimental stability work of Becker and Kaye [13] and Bjorklund and Kays [20] with gases no significant effect of gravity was observed. Further we might note that in stability problems of this type the critical Taylor number is rather insensitive to variations in the basic velocity and temperature profiles. Thus we will ignore the apparently weak convective motions due to gravity in the present analysis. The effect of essentially a radial gravity field on various curved flows has been considered by Görtler [21] and Kirchgässner [22].

In addition to the three momentum equations and the continuity equation we must now include an energy equation and an equation of state in our analysis. We shall assume that our fluid has constant thermal conductivity, specific heat, and viscosity*, and we will neglect viscous dissipation and energy associated with change in pressure in the energy equation. With these assumptions the basic azimuthal velocity distribution is given by Eq. (7), and the temperature distribution Θ is given by

$$\Theta(r) = T_1 - (T_2 - T_1)\frac{\ln(r/R_1)}{\ln \eta} \tag{24}$$

where T_1 and T_2 are the temperatures of the inner and outer cylinders respectively. As an equation of state we assume $\rho = \rho_0[1-\alpha(\Theta - \Theta_0)]$ where Θ_0 is a reference temperature and α is the bulk coefficient of thermal expansion.

We now consider rotationally symmetric disturbances of this steady

---

*Thus our analysis is primarily limited to gases with small temperature differences or liquids with only slight viscosity variation with temperature. The corresponding small gap problem for liquids with variable viscosity has been considered by Walowit [23].

velocity and temperature distribution. A small disturbance $\theta(r,z,t)$ in the temperature will lead to a disturbance in the density. For small temperature variations, the variation in density will be small and we will assume that the density is a constant ρ, except when multiplied by the centrifugal acceleration term V^2/r. Here the density disturbance is given by $-\rho\alpha\theta'$. This is the classic Boussinesq approximation. The disturbances can be expressed in the form given in Eq. (2), that is, $\theta' = \theta(r)\, e^{\sigma t} \cos\lambda z$ and we obtain, using the principle of exchange of stabilities, the following eighth order system of equations

$$
\begin{aligned}
&\nu (DD^* - \lambda^2)^2 u = \lambda^2 \left\{ 2\Omega v - \frac{\alpha}{r} V^2 \theta \right\} \\
&\nu (DD^* - \lambda^2) v = (2A)\, u \qquad\qquad (25) \\
&\kappa (D^*D - \lambda^2) \theta = (D\Theta)\, u
\end{aligned}
$$

with the boundary conditions

$$u = Du = v = \theta = 0 \qquad\qquad (26)$$

at $r = R_1$ and R_2. Here κ is the thermal diffusivity. In dimensionless variables Eqs. (25) take the form

$$
\begin{aligned}
&(DD^* - a^2)^2 u = - a^2 T \left\{ g(x)\, v + N \zeta \left[g(x) \right]^2 \theta \right\} \\
&(DD^* - a^2) v = u \qquad\qquad (27) \\
&(D^*D - a^2) \theta = u/\zeta
\end{aligned}
$$

where $x, a, g(x), \zeta, T, D$, and D^* are defined by Eqs. (9), (12), (13) and (14).

Also u has been replaced by $(2Ad^2/\nu)^{-1}u$ and θ has been replaced by $\left[Pr(T_2-T_1)/2AR_2 \ln(1/\eta)\right]\theta$, where $Pr = \nu/\kappa$ is the Prandtl number. Here

$$N = \frac{Pr\,\alpha(T_2-T_1)}{4A_1 \ln\eta} \tag{28}$$

is the ratio of a Rayleigh number, $Ray = \alpha(T_2-T_1)d^3 R_{LM}\Omega_1^2/\nu\kappa$ with $R_{LM} = -d/\ln\eta$, based on the centrifugal acceleration $R_{LM}\Omega_1^2$ and the Taylor number T.

In the small-gap limit, $\eta \to 1$, $N \to Pr\,\alpha(T_2-T_1)/2(1-\mu)$. Moreover, as $\eta \to 1$, $\zeta \to 1$, $D^* \to D$; hence it follows from the last of Eqs. (27) and the boundary conditions (26) that $\theta = v$. Consequently the eighth order system of equations reduces to a sixth order system. This problem has been considered by Yih [10] and Becker and Kays [12] for $\mu > 0$ and by Lai [11] for $\mu \gtrless 0$.

The eigenvalue problem,

$$F(\eta, \mu, a, N, T) = 0$$

defined by Eqs. (27) and the boundary conditions (26) at $x = \pm\frac{1}{2}$, can be solved by the Galerkin method described previously. The expansion functions for θ are the same as those for v. Computations have been carried out using two and three terms in the series (9x9 determinant) for $\eta \to 1$, $\mu = 0$; and $\eta = \frac{1}{2}$, $\mu = 0$, and 0.2 for various values of N, and for $\eta = \frac{1}{2}$, $\mu = \frac{1}{4}$ for various values of Ray. The results are tabulated in Table 3 and are shown graphically in Fig. 5.

For the isothermal case $N = 0$ the results are within 1% of the results of Chandrasekhar [2,3]; and for $N \neq 0$ but $\eta \to 1$ they are within 1% of those given by Lai [11]. Further, the maximum change in the critical Taylor number between the two and three term approximations is of the order of 3%.

The critical values of a given in Table 3 have only been determined up to ± 0.04.

CONCLUSIONS

The present results indicate that the Galerkin method in conjunction with simple polynomial expansion functions can yield accurate results for a variety of stability problems. Further, once the integrals corresponding to such operators as $(DD^* - a^2)^2\, u$, $(DD^* - a^2)\, v$, and $(D^*D - a^2)\, \theta$ are evaluated a number of different stability problems can be treated with only minor additional computations.

In Fig. 1 the variation of $T\eta^2$ with η for $\mu = 0$ is shown. Also the theoretical results of Chandrasekhar and Elbert [5] and Kirchgässner [7] as well as the experimental results of Taylor [1], Lewis [24], Donnelly [25] and Caldwell and Donnelly [26] are indicated. In general the agreement is excellent. The hysteresis effect indicated by Lewis appears to be incorrect. See Caldwell and Donnelly. It is interesting to note that Taylor's correction formula for gap size is indeed very accurate for $\eta \geqslant 0.8$ but does begin to deviate fairly rapidly from the present results for $\eta < 0.8$. From a practical point of view the present results indicate the correction in T in comparing small-gap theoretical ($\eta = 1$) values with small-gap experimental (η near 1) results. Thus for $\eta = 1$, $\mu = 0$, $T = 3391$ and for $\eta = .95$ (a reasonable value for small-gap experiments), $\mu = 0$, $T = 3511$ which is a change of 3.5%. As mentioned earlier for $\mu = 0$, $T\eta^2 = (2\,\Omega_1 R_1^2/\nu)^2(1+\eta^2)$ approaches a value of approximately 500 as $\eta \rightarrow 0$.

In Fig. 2 the variation of $T\eta^2$ with η for $\mu = 0.2$, 0, -0.125, and -0.25 is shown. Finally the regions of stability in a $(\Omega_1 R_1 d/\nu)(d/R_1)^{\frac{1}{2}}$ vs $(\Omega_2 R_1 d/\nu)(d/R_1)^{\frac{1}{2}}$ plane are depicted in Fig. 3 for $\eta = 1.0$, 0.7, and 0.5.

It is well known for the small-gap problem, $\eta \rightarrow 1$, that even if $\Omega(r)$ is replaced by its average value accurate results are obtained for $\mu \geq 0$. Lin [27] has suggested that it should also be possible to do this for the finite gap problem. However, some care must be taken in choosing a suitable average. It is natural to consider the velocity rather than the angular velocity, and to replace $V(r)$ by its average over the gap, V_{av}. To obtain a suitable average angular velocity, V_{av} must be divided by a suitable length scale independent of μ. The requirement that the average angular velocity be the same as the angular velocities of the cylinders in the limit $\mu \rightarrow 1$, leads to the choice of R_o for this length. Thus $g(x)$ in Eq. (10) is replaced by $A_1 + B_1/(\eta_o \eta_{lm})$ where $\eta_o = (1+\eta)/2$ and $\eta_{LM} = (1-\eta)/(-\ln \eta)$; and we can anticipate that

$$T(A_1 + \frac{B_1}{\eta_o \eta_{LM}}) = T_{av} \qquad (29)$$

should depend only upon η for $\mu \geq 0$. Actual computations show that this dimensionless parameter gives the correct value of T within 2% for a range of η from 0.2 to 1.0 and for $\mu \geq 0$. In addition, it was found that the critical value of T_{av} is relatively insensitive to variations in η. By using its value at $\eta = 1$, namely 1708, we obtain from Eq. (29) after substituting for A_1 and B_1 from Eqs. (12) the extremely simple formula for the critical value of T

$$T = \frac{1708(1-\eta^2)}{\mu - \eta^2 + (1-\mu)\eta^2/\eta_o \eta_{LM}} \qquad (30)$$

which is correct within 4% in the range $0.5 \leq \eta \leq 1$, $\mu \geq 0$.*

*After this work was completed the authors learned that Rintel and Lieber [33] have also suggested an averaging process for computing the critical Taylor number for the finite gap. Their averaging is considerable more complicated and does not appear to yield any more accurate results.

For the Dean stability problem the variation of $(V_m d/\nu)(d/R_2)^{\frac{1}{2}}$ with η is shown in Fig. 4. Of particular interest mathematically is the non-monotonic behavior of P with an increasing number of terms in the series expansion. For example for $\eta = 1$ and $a = 3.96$ the result 36.86 for $M = 2$ is closer to what we might consider the "exact" value 35.94 than the result 37.26 for $M = 3$. At first sight these results seemed surprising and were consequently checked by hand computation. However, if we bear in mind that these are nonself-adjoint eigenvalue problems we realize that we cannot expect necessarily monotonic behavior of P with increasing M. Indeed, this result shows that the well known result for self-adjoint, positive definite eigenvalue problems that the approximation to the eigenvalue is improved with increasing M does not hold in general for nonself-adjoint eigenvalue problems.

Finally, we consider the effect of a radial temperature gradient on the stability of Couette flow. As shown in Fig. 5, positive and negative temperature gradients are destabilizing and stabilizing respectively. For μ not near η^2 it is difficult in practice to achieve values of N, without violating the assumptions we have made, large enough to change significantly the critical Taylor number from that predicted by isothermal theory.

However, A_1 will be small for μ near η^2, and hence it will be possible to have large values of N. In this case the Rayleigh number plays an important role. A graph of the critical Taylor number vs. Ray number for $\mu \to \eta^2$ $(N = \frac{1}{2})$ is given in Fig. 6. As can be seen from this figure, for sufficiently large Ray numbers it is possible for the flow to be unstable even though T is negative, that is $\mu > \eta^2$. This was first pointed out by Yih [10] and Becker and Kaye [12] for the small-gap problem. Further, Yih observed for the small-gap problem as is the case here, that von Karman's condition based

on an inviscid fluid (that the product of the density ρ and the square of the circulation Γ increasing outward is a necessary and sufficient condition for stability) no longer holds when viscosity is taken into account. This can be seen from the plot of T vs. N for $\mu \to \eta^2$ $(n = \frac{1}{2})$, Fig. 7. The points in the lower left hand quadrant correspond to ρ decreasing and Γ increasing radially. Thus for suitable Prandtl numbers one can find flows which are unstable even though $\rho\Gamma^2$ increases outward and the flow is stable according to the inviscid theory.

In a manner similar to that used in deriving Eq. (30), and making use of the rather insensitive dependence of the differential operators on η for $\frac{1}{2} \leq \eta \leq 1$ it is possible to derive an "average-type" stability criterion for the thermal problem. For $\mu \geq 0$, $\frac{1}{2} \leq \eta \leq 1$ this relation is given by

$$T_{av} + (Ray)_{av} = 1708 \tag{31}$$

where T_{av} is defined by Eq. (29) and

$$(Ray)_{av} = Ray\left(A_1 + \frac{B_1}{\eta_o \eta_{LM}}\right)^2 . \tag{32}$$

This result reduces to that given by Becker and Kaye [12] for $\eta \to 1$.

ACKNOWLEDGMENT

The authors would like to express their appreciation to Dr. Trevor Stuart for his helpful comments.

REFERENCES

1. G. I. Taylor, "Stability of a Viscous Liquid Contained Between Two Rotating Cylinders," Philosophical Transactions, A, vol. 223, 1923, pp. 289-343.

2. S. Chandrasekhar, "Hydrodynamic and Hydromagnetic Stability," Oxford, London, England, 1961.

3. S. Chandrasekhar, "The Stability of Viscous Flow Between Rotating Cylinders," Proceedings of the Royal Society, London, England, series A, vol. 246, 1958, pp. 301-311.

4. S. Chandrasekhar, "Adjoint Differential Systems in the Theory of Hydrodynamic Stability," Journal of Mathematics and Mechanics, vol. 10, 1961, pp. 683-690.

5. S. Chandrasekhar and D. Elbert, "The Stability of Viscous Flow Between Rotating Cylinders II," Proceedings of the Royal Society, London, England, series A, vol. 268, 1962, pp. 145-152.

6. H. Witting, "Über den Einflub der Stromlinienkrümmung auf die Stabilität Laminarer Strömungen," Archive for Rational Mechanics and Analysis, vol. 2, 1958, pp. 243-283.

7. K. Kirchgässner, "Die Instabilität der Strömung swischen swei rotierenden Zylindern gegenünber Taylor-Wirbeln für beliebige Spaltbreiten," Zeitschrift für Angewandte Mathematik und Physik, vol. 12, 1961, pp. 14-29.

8. W. R. Dean, Fluid Motion in a Curved Channel," Proceedings of the Royal Society, London, England, series A, vol. 121, 1928, pp. 402-420.

9. B. Meister, "Das Taylor-Deansche Stabilitätsproblem für beliebige Spaltbreiten," Zeitschrift für Angewandte Mathematik und Physik, vol. 13, 1962, pp. 83-91.

10. Chia-Shun Yih, "Dual Role of Viscosity in the Instability of Revolving Fluids of Variable Density," The Physics of Fluids, vol. 4, 1961, pp. 806-811.

11. Wei Lai, "Stability of a Revolving Fluid with Variable Density in the Presence of a Circular Magnetic Field," The Physics of Fluids, vol. 5, pp. 560-566.

12. K. M. Becker and J. Kaye, "The Influence of a Radial Temperature Gradient on the Instability of Fluid Flow in an Annulus with an Inner Rotating Cylinder," Trans. ASME, series C, vol. 84, 1962, pp. 106-110.

13. K. M. Becker and J. Kaye, "Measurements of Diabatic Flow in an Annulus with an Inner Rotating Cylinder," Journal of Heat Transfer, Trans. ASME, series C, vol. 84, 1962, pp. 97-105.

14. P. Roberts, private communication.

15. W. Reid, "On the Stability of Viscous Flow in a Curved Channel," Proceedings of the Royal Society, London, England, series A, vol. 244, 1958, pp. 186-198.

16. G. Hämmerlin, "Die Stabilität der Strömung in einem gekrümmten Kanal," Archive for Rational Mechanics and Analysis, vol. 1, 1958, pp. 212-224.

17. E. R. Eckert and R. M. Drake, "Heat and Mass Transfer," second edition, McGraw-Hill Book Co., New York, 1959, Chapter 11.

18. Chen-Ya Liu, W. K. Mueller and F. Landis, "Natural Convection Heat Transfer in Long Horizontal Cylindrical Annuli," International Developments in Heat Transfer, ASME, vol. 5, 1961, pp. 976-994.

19. L. Crawford and R. Lemlich, "Naturel Convection in Horisontal Concentric Cylindrical Annuli," Industrial and Engineering Chemistry Fundamentals, vol. 1, 1961, pp. 260-264.

20. I. S. Bjorklund and W. M. Kays, "Heat Transfer between Concentric Rotating Cylinders," Trans. ASME, vol. 81, 1959, pp. 175-186.

21. H. Görtler, "Über eine Analogie zwischen den Instabilitäten Laminarer Grenzschichströmungen an Konkaven Wänden und an Erwärmten Wänden," Ingenieur-Archiv, vol. 28, 1959, pp. 71-78.

22. K. Kirchgässner, "Einige Beispiele Stabilitätstheorie von Strömungen an Konkaven und Erwärmten Wänden," Ingenieur-Archiv, vol. 31, 1962, pp. 115-124.

23. J. Walowit, "The Stability of Couette Flow between Rotating Cylinders in the Presence of a Radial Temperature Gradient," Ph.D. Thesis, Rensselaer Polytechnic Institute, Troy, N.Y., 1963.

24. J. Lewis, "An Experimental Study of the Motion of a Viscous Liquid Contained Between Two Coaxial Cylinders," Proceedings of the Royal Society, London, England, series A, vol. 117, 1928, pp. 388-407.

25. R. J. Donnelly, "Experiments on the Stability of Viscous Flow Between Rotating Cylinders I. Torque Measurements," Proceedings of the Royal Society, series A, vol. 246, 1958, pp. 312-325.

26. D. Caldwell and R. J. Donnelly, "On the Reversibility of the Transition past Instability in Couette Flow," Proceedings of the Royal Society, series A, vol. 267, 1962, pp. 197-205.

27. C. C. Lin, "The Theory of Hydrodynamic Stability," Cambridge University Press, Cambridge, England, 1955, pp. 18-22.

28. R. C. DiPrima, "Application of the Galerkin Method to Problems in Hydrodynamic Stability," vol. 13, 1955, pp. 55-62.

29. F. C. Haas and A. H. Nissan, "Experimental Heat Transfer Characteristics of a Liquid in Couette Motion and with Taylor Vortices," Proceedings of the Royal Society, London, England, series A, vol. 261, 1961, pp. 215-226.

30. C. Y. Ho, J. N. Nardacci and A. H. Nissan, "Heat Transfer Characteristics of Fluids Moving in a Taylor System of Vortices," to be published in the American Institute of Chemical Engineers Journal.

31. U. H. Kurzweg, "Magnetohydrodynamic Stability of Curved Viscous Flows," Princeton University, Department of Physics, Technical Report II-29, 1961.

32. D. L. Harris and W. H. Reid, "On Orthogonal Functions which Satisfy Four Boundary Conditions I. Tables for Use in Fourier-type Expansions," Astrophysics Journal Supplement, Series 3, 1958, pp. 429-47.

33. L. Rintel and P. Lieber, "On the Stability of Viscous Flow Between Rotating Cylinders; Finite Gap," Report No. MD-63-1, Institute of Engineering Research, University of California, Berkeley, Calif., Feb. 1963.

Table 1. Critical Taylor numbers and corresponding values of a for various assigned values of $\mu = \Omega_2/\Omega_1$ and $\eta = R_1/R_2$. Subscripts denote the number of terms used in the approximating series. The results given in column C are those given by Chandrasekhar [2] and Chandrasekhar and Elbert [5].

η	μ	a	$(T\eta^2)_2$	$(T\eta^2)_3$	$(T\eta^2)_4$	C	T_2/T_3	T_3/T_4	T_2/T_4
1.00	1.00	3.12	1750.0	1708.5	1708.4	1708.1	1.024	1.000	1.024
	0.5	3.12	2331.4	2276.4	2276.1	2275.3	1.024	1.000	1.024
	0.25	3.12	2792.7	2727.1	2725.9	2725.3	1.024	1.000	1.025
	0	3.12	3475.0	3394.6	3391.3	3390.3	1.023	1.001	1.025
	-0.125	3.13	3952.9	3863.0	3856.6	----	1.023	1.012	1.025
	-0.25	3.13	4575.8	4474.4	4463.4	4462.6	1.023	1.003	1.025
	-0.50	3.20	6589.4	6462.7	6418.4	6417.1	1.020	1.007	1.026
	-0.70	3.34	9718.1	9598.3	9435.6	9433.0	1.012	1.017	1.030
	-0.90	3.70	15537	15559	14952	14940	0.999	1.042	1.039
	-1.00	4.00	19644	19745	18703	18680	0.995	1.055	1.050
0.95	0.9025	3.12	1663.5	1624.2	1624.2	----	1.024	1.000	1.024
	0.70	3.12	1869.3	1825.3	1825.1		1.024	1.000	1.024
	0.50	3.12	2128.7	2078.7	2078.4		1.024	1.000	1.024
	0.20	3.12	2684.9	2622.9	2621.6		1.024	1.001	1.024
	0	3.13	3245.4	3172.4	3168.8		1.023	1.001	1.024
	-0.125	3.14	3727.2	3645.5	3638.8		1.022	1.002	1.024
	-0.25	3.15	4368.1	4276.1	4263.4		1.021	1.003	1.025

Table 1. (Continued)

η	μ	a	$(T\eta^2)_2$	$(T\eta^2)_3$	$(T\eta^2)_4$	c	T_2/T_3	T_3/T_4	T_2/T_4
0.90	0.81	3.12	1579.0	1541.9	1541.9		1.024	1.000	1.024
	0.50	3.12	1934.9	1890.1	1889.6		1.024	1.000	1.024
	0.20	3.12	2470.9	2415.2	2413.7		1.023	1.001	1.024
	0	3.13	3024.7	2958.9	2955.0		1.022	1.001	1.024
	-0.125	3.14	3511.7	3438.1	3430.8		1.021	1.002	1.024
	-0.25	3.15	4175.0	4092.8	4078.4		1.020	1.004	1.024
0.80	0.64	3.12	1415.7	1383.6	1383.3		1.023	1.000	1.023
	0.40	3.12	1710.8	1673.9	1672.4		1.022	1.001	1.023
	0.20	3.13	2068.4	2024.6	2022.8		1.021	1.001	1.023
	0	3.13	2609.6	2557.8	2553.4		1.020	1.002	1.022
	-0.125	3.16	3112.8	3055.3	3046.8		1.019	1.003	1.022
	-0.25	3.17	3840.5	3778.1	3759.9		1.016	1.005	1.022
0.70	0.49	3.12	1260.2	1233.5	1233.0		1.022	1.000	1.022
	0.35	3.13	1440.2	1410.6	1409.7		1.021	1.000	1.022
	0.20	3.13	1699.2	1665.8	1664.2		1.020	1.001	1.021
	0	3.14	2229.5	2190.4	2185.9		1.018	1.002	1.020
	-0.125	3.15	2760.2	2718.7	2708.5		1.016	1.004	1.019
	-0.25	3.19	3597.0	3556.7	3532.1		1.012	1.007	1.018
	-0.50	3.56	----	7821.9	7586.6		---	1.031	---

Table 1. (Continued)

η	μ	a	$(T\eta^2)_2$	$(T\eta^2)_3$	$(T\eta^2)_4$	C	T_2/T_3	T_3/T_4	T_2/T_4
0.60	0.36	3.13	1112.9	1091.5	1091.0		1.020	1.001	1.020
	0.25	3.13	1272.9	1249.7	1248.6		1.019	1.001	1.019
	0.10	3.14	1581.5	1555.8	1553.2		1.017	1.002	1.018
	0	3.15	1883.5	1856.6	1851.5		1.015	1.003	1.017
	-0.125	3.17	2466.9	2436.6	2424.8		1.015	1.005	1.017
	-0.25	3.24	3512.6	3488.2	3453.2		1.007	1.010	1.017
0.50	0.25	3.14	973.88	958.35	957.22	957.8	1.017	1.001	1.017
	0.20	3.14	1054.4	1038.5	1037.0		1.016	1.001	1.017
	0.10	3.15	1263.2	1246.2	1243.7		1.015	1.002	1.016
	0	3.16	1574.7	1554.5	1549.6	1550	1.013	1.003	1.016
	-0.125	3.19	2250.4	2230.3	2216.4	2219	1.009	1.006	1.015
	-0.25	3.32	3774.5	3752.0	3693.9	3701	1.006	1.015	1.022
	-0.50	4.78	----	13553	13323	13022	---	1.017	---
0.40	0.16	3.15	845.74	833.21	831.81		1.015	1.002	1.017
	0.08	3.16	1025.6	1011.5	1009.1		1.014	1.002	1.016
	0	3.17	1298.6	1283.5	1279.1		1.012	1.003	1.015
	-0.125	3.24	2253.0	2164.2	2146.5		1.041	1.008	1.049
	-0.25	3.76	5179.1	4942.5	4830.6		1.048	1.023	1.072
0.30	0.09	3.19	728.00	716.56	714.81		1.016	1.002	1.018
	0.05	3.20	846.17	833.10	830.70		1.016	1.003	1.019

Table 1. (Continued)

η	μ	a	$(T\eta^2)_2$	$(T\eta^2)_3$	$(T\eta^2)_4$	T_2/T_3	T_3/T_4	T_2/T_4
0.30	0	3.20	1058.7	1043.1	1040.0	1.015	1.004	1.018
	-0.125	3.44	2623.8	2525.4	2495.2	1.039	1.012	1.051
0.20	0.04	3.22	624.36	609.12	606.72	1.025	1.004	1.029
	0	3.23	849.24	833.40	829.14	1.019	1.005	1.024
0.10	0.01	3.29	541.11	513.87	509.10	1.053	1.009	1.062
	0	3.30	673.59	657.09	650.00	1.025	1.011	1.036

Table 2. Critical values of $R(d/R_2)^{\frac{1}{2}} = (V_m d/\nu)(d/R_1)^{\frac{1}{2}}$ and corresponding values of a for various assigned values of η. Subscripts denote the number of terms used in the approximating series.

η	a	$[R(d/R_2)^{\frac{1}{2}}]_2$	$[R(d/R_2)^{\frac{1}{2}}]_3$	$[R(d/R_2)^{\frac{1}{2}}]_4$	R_2/R_3	R_3/R_4	R_2/R_4
1.0	3.96	36.84	37.31	35.90	0.989	1.039	1.025
0.95	4.02	37.19	37.70	36.26	0.986	1.040	1.025
0.90	4.06	37.72	38.30	36.79	0.985	1.041	1.024
0.80	4.16	38.82	39.51	37.88	0.983	1.043	1.024
0.70	4.24	40.11	40.96	39.19	0.979	1.045	1.023
0.60	4.32	41.64	42.73	40.79	0.975	1.046	1.021
0.50	4.41	43.47	44.91	42.79	0.968	1.050	1.017
0.40	4.46	45.65	47.70	45.18	0.957	1.053	1.011
0.30	4.51	48.47	51.47	48.75	0.942	1.056	0.995
0.20	4.57	51.72	57.00	53.60	0.908	1.063	0.964
0.10	4.64	57.03	65.91	60.92	0.865	1.082	0.936

Table 3. Critical Taylor numbers and corresponding values of a for various assigned values of η, μ, and N or Ray. Subscripts denote the number of terms used in the approximating series.

η	μ	a	N	Ray	T_2	T_3	T_2/T_3
1.0	0	3.12	1.0		2223.4	2177.3	1.0212
			0.5		3712.1	2653.3	1.3990
			0		3474.7	3394.6	1.0235
			-0.5		4829.4	4705.9	1.0262
			-0.75		5992.0	5826.3	1.0284
			-1.0		7874.9	7631.9	1.0318
0.5	0	3.15	1.0		4659.9	4609.0	1.0110
			0.5		5356.9	5295.3	1.0116
			0		6296.4	6218.2	1.0126
			-0.5		7628.7	7522.2	1.0142
			-0.75		8526.3	8397.3	1.0154
			-1.0		9656.6	9494.8	1.0170
0.5	0.2	3.15	1.0		2895.2	2852.3	1.0150
			0.5		3435.4	3382.8	1.0155
			0		4221.9	4154.2	1.0163
			-0.5		5471.3	5376.8	1.0176
			-0.75		6417.1	6300.0	1.0186
			-1.0		7750.7	7598.0	1.0201

Table 3. (Continued)

η	μ	a	N	Ray	T_2	T_3	T_2/T_3
0.5	0.25	3.15		16000	-3996.7	-4041.2	.989
				12000	-2010.2	-2058.1	.977
				7828		0	
				4000	19375	1879.2	1.031
				0	3898.3	3833.6	1.017
				-4000	5850.6	5778.5	1.012
				-7000	7309.1	7231.1	1.011

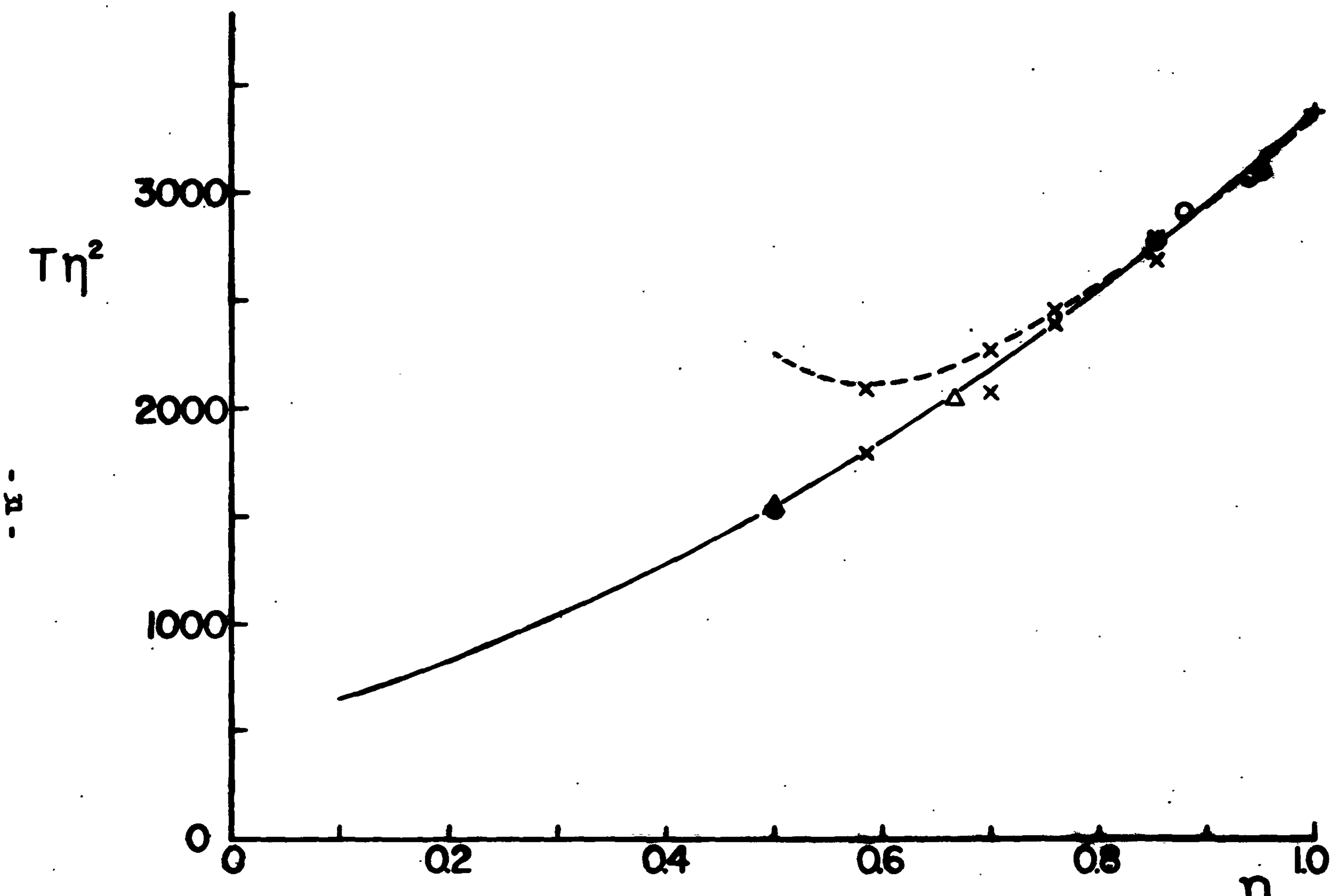

Fig. 1. The variation of $T\eta^2$ with η for $\mu = 0$. The experimental results of Lewis [24], Taylor [1], and Donnelly [25] and Caldwell and Donnelly [26] are indicated by ×, ○, and ● respectively. The theoretical results of Chandrasekhar and Elbert [5] and Kirchgässner [7] are denoted by + and △ respectively. The finite-gap approximation by Taylor [1] is denoted by -------.

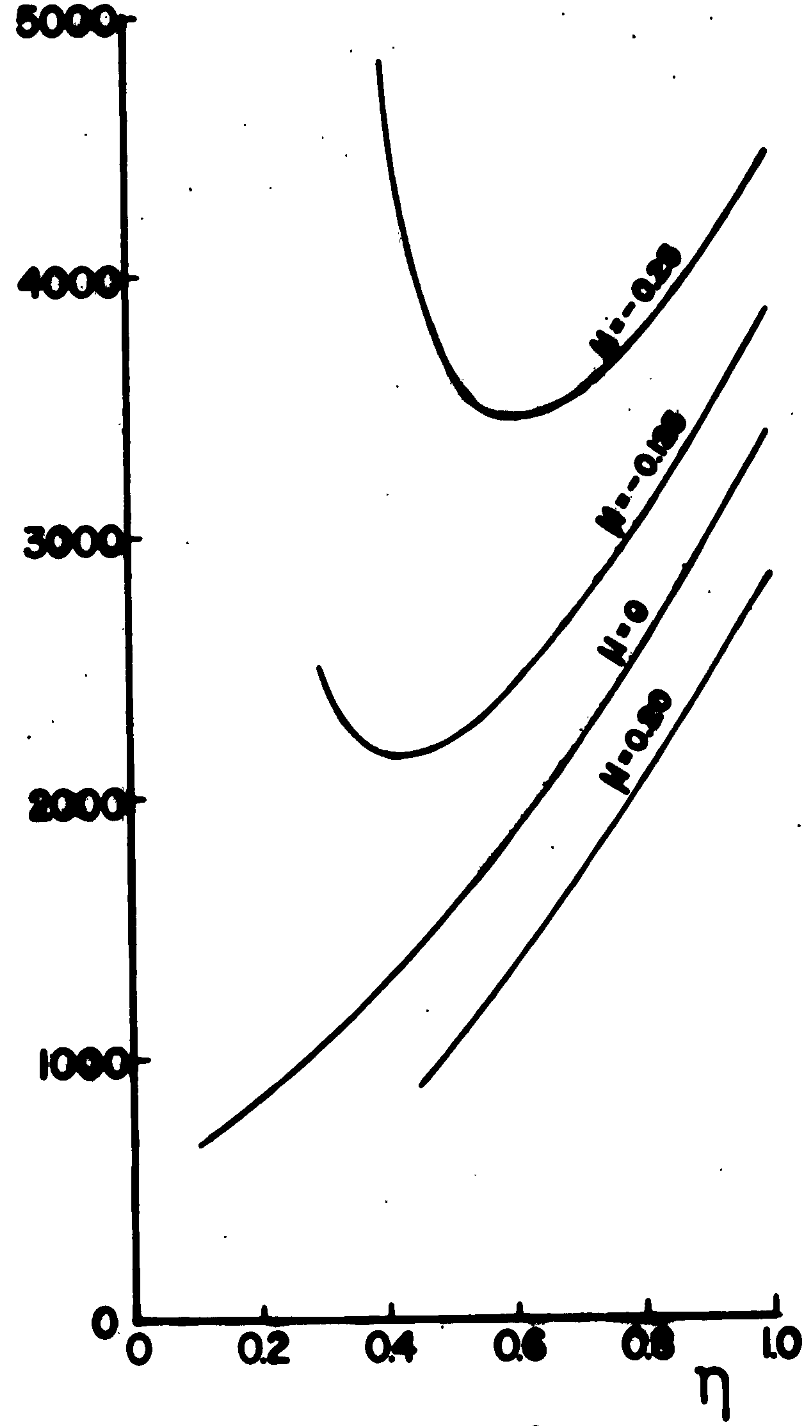

Fig. 2. The variation of $T\eta^2$ with η for $\mu = 1/5$
0, - 1/8, - 1/4.

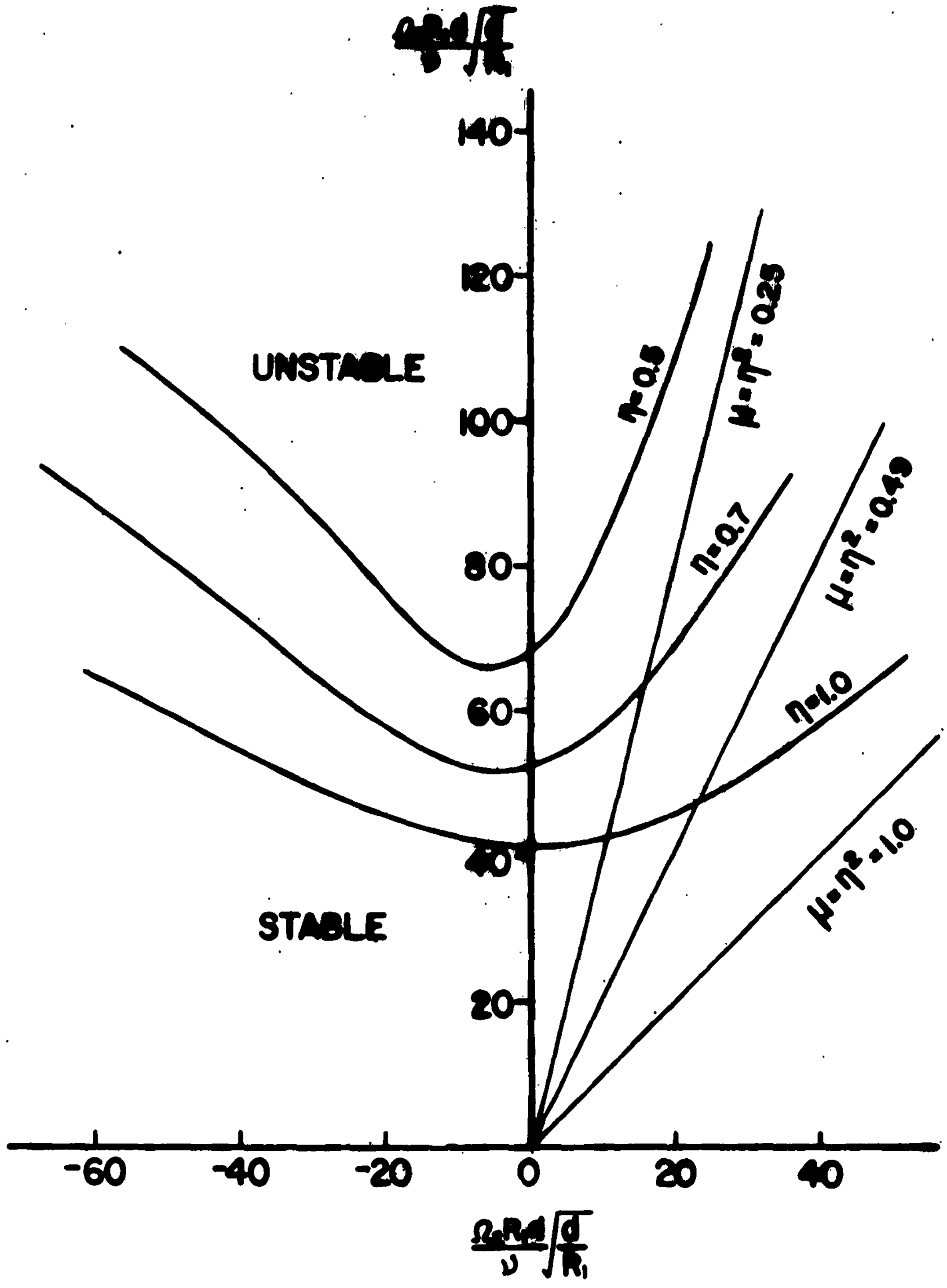

Fig. 3. The stable and unstable regions in a $(\Omega_1 R_1 d/\nu)(d/R_1)^{\frac{1}{2}}$ vs $(\Omega_2 R_1 d/\nu)(d/R_1)^{\frac{1}{2}}$ plane for various values of η.

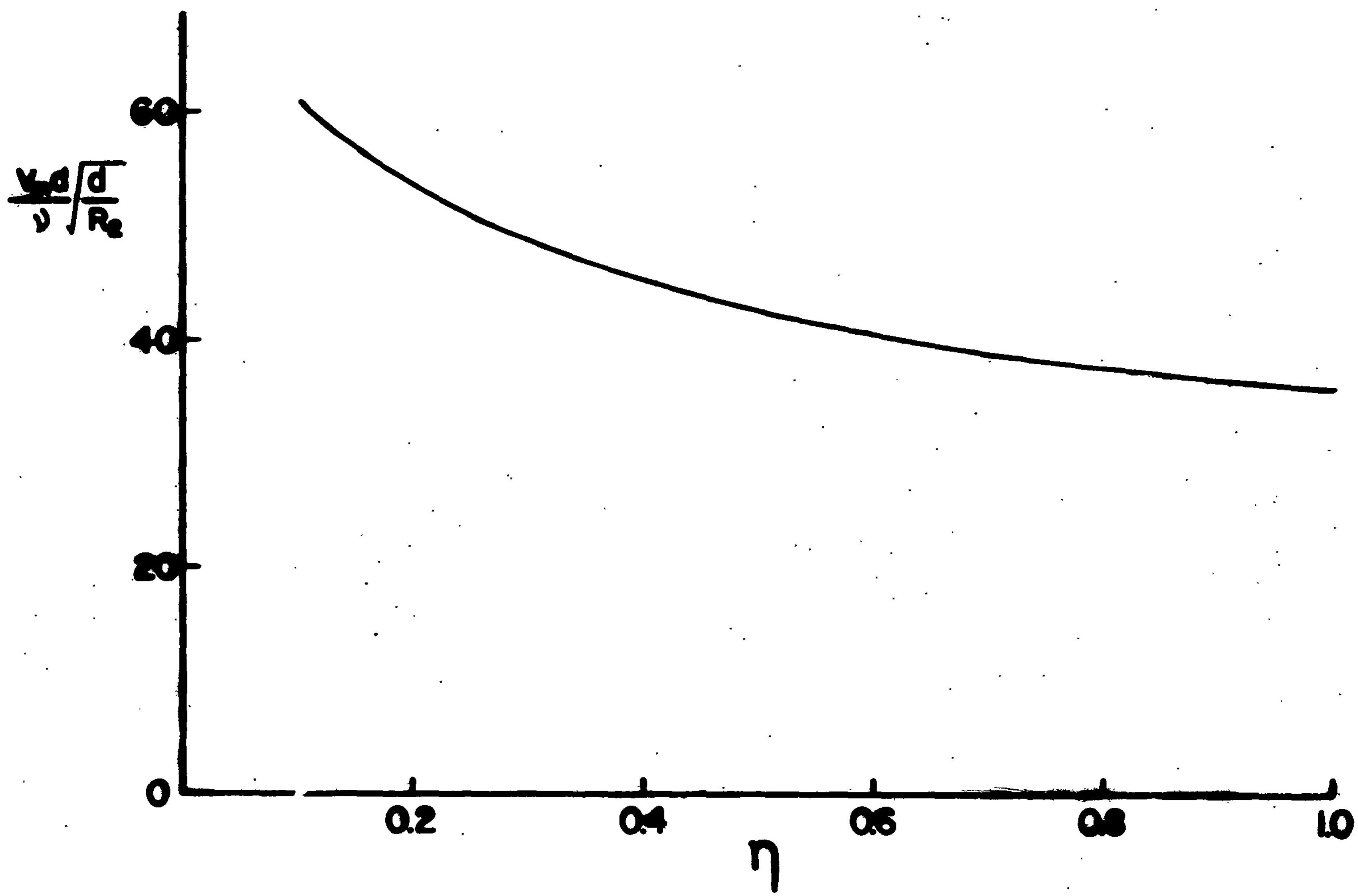

Fig. 4. The variation of $(V_m d/\nu)(d/R_2)^{\frac{1}{2}}$ with η.

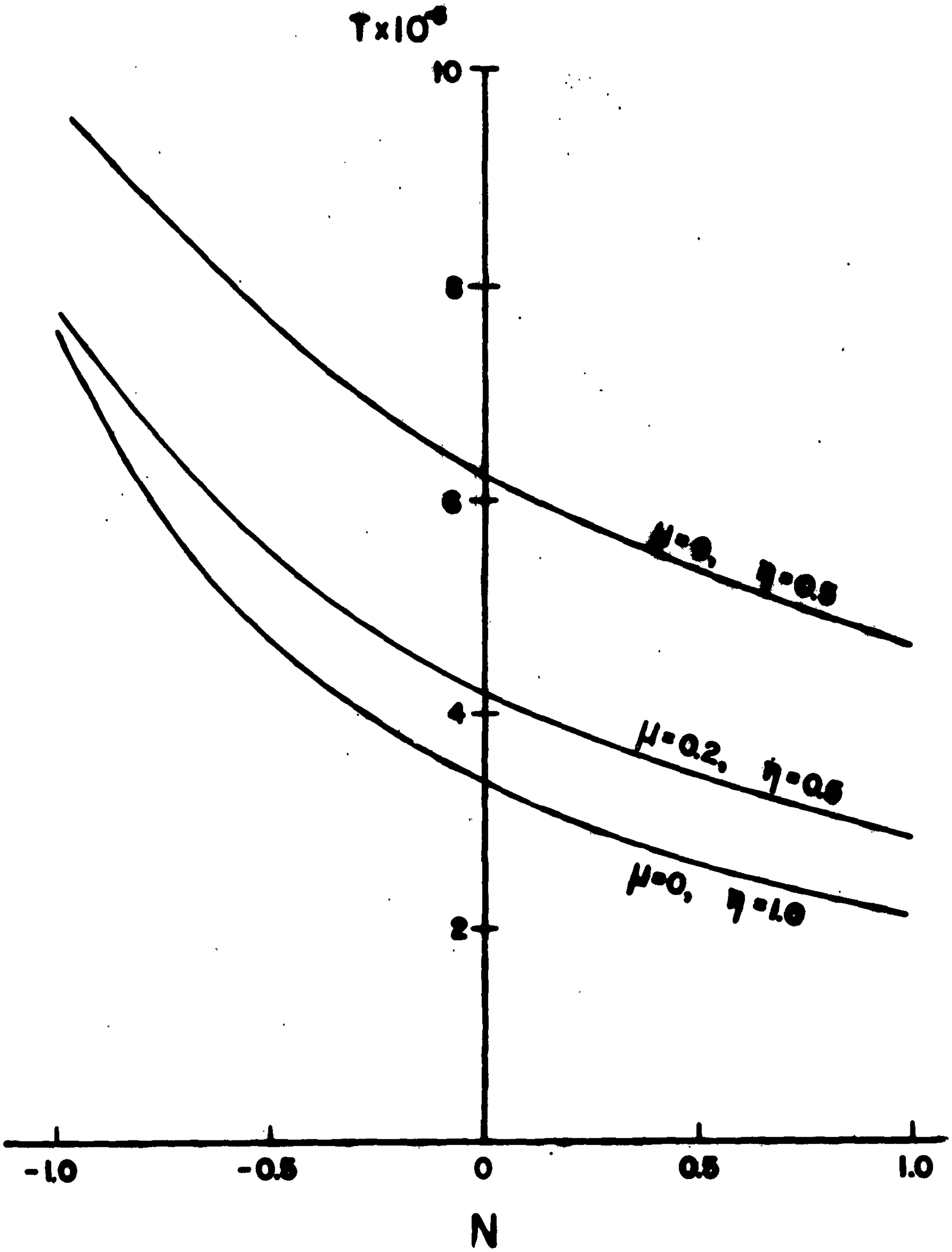

Fig. 5. The variation of the Taylor number T with N for assigned values of μ and η.

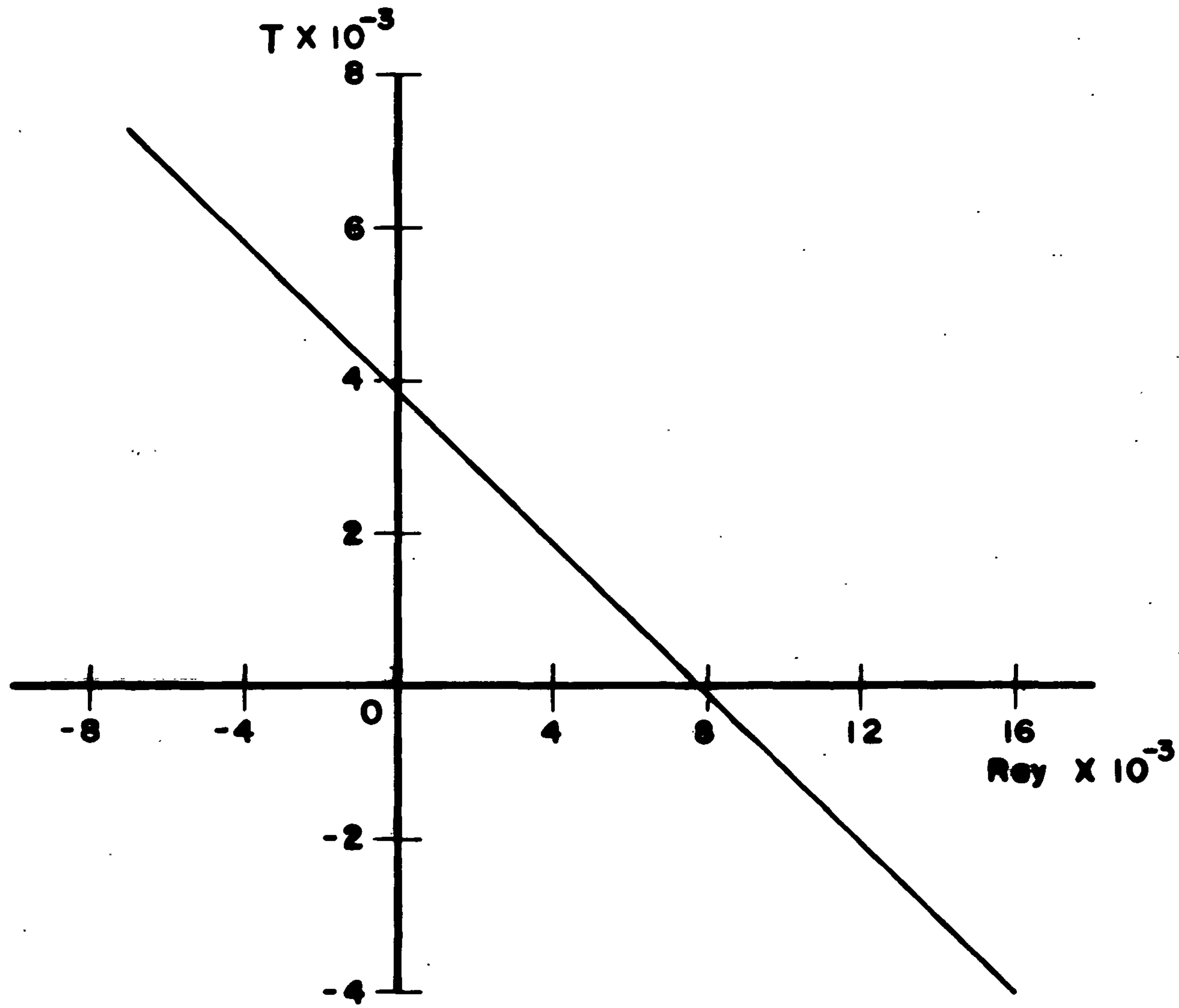

Fig. 6. The variation of the Taylor number T with the Rayleigh number Ray for $\mu \to \eta^2$, $\eta = \frac{1}{2}$.

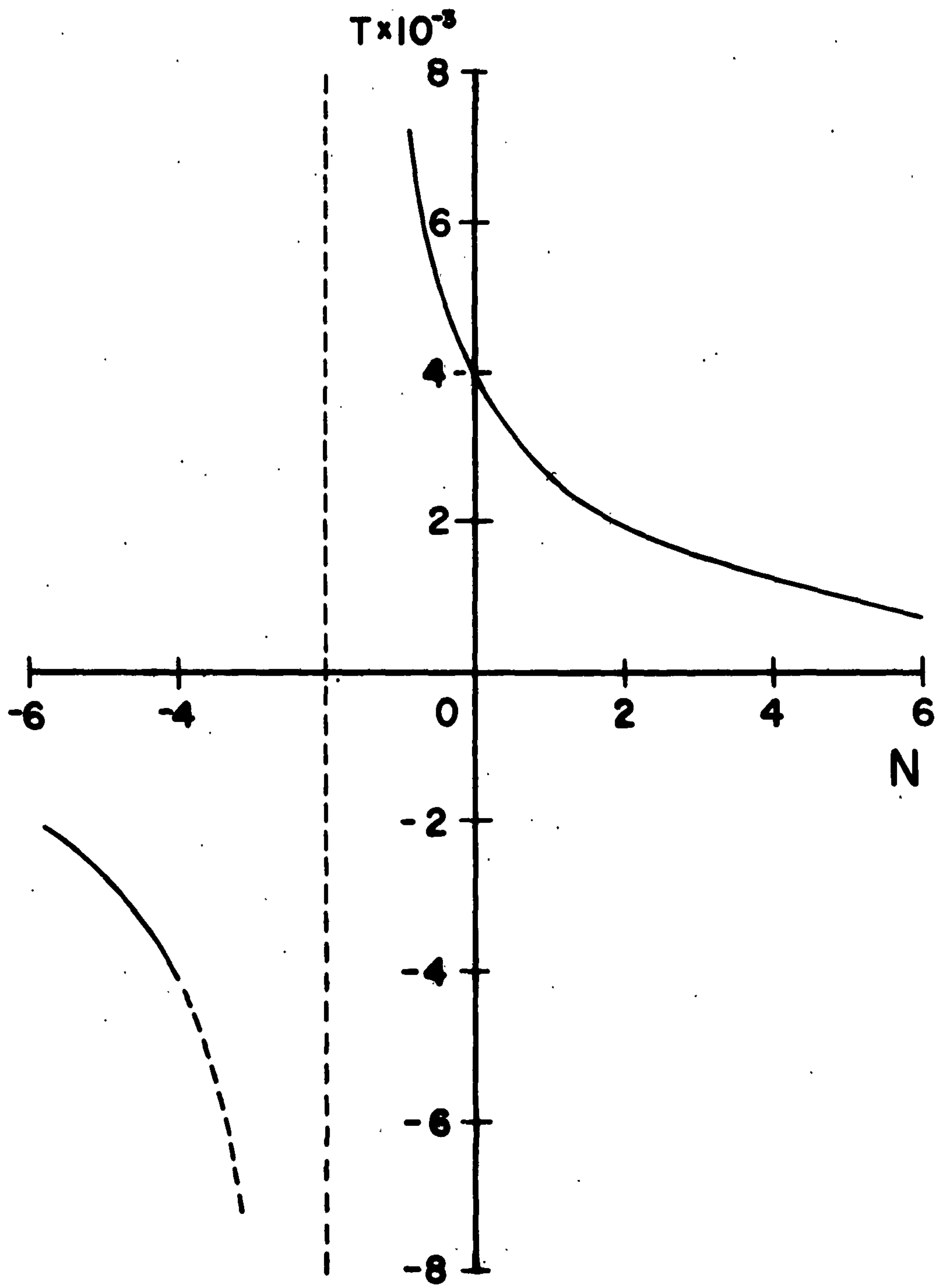

Fig. 7. The variation of the Taylor number T with N for $\mu \to \eta^2$, $\eta = \frac{1}{2}$.

APPENDIX

All of the integrals that are needed to use the Galerkin method with the polynomial functions

$$u_n(x) = (x^2 - \tfrac{1}{4})^2\, x^{n-1} \quad , \quad v_n(x) = \theta_n(x) = (x^2 - \tfrac{1}{4})\, x^{n-1} \tag{A1}$$

can be expressed as linear combinations of the following three functions:

$$S(m,n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^m (x^2 - \tfrac{1}{4})^n\, dx = \begin{cases} 0 & , \ m \text{ odd} \\ \dfrac{(-1)^n\, n!\, (\frac{1}{2})^{m+n}}{(m+1)(m+3)\ldots(m+2n+1)} & , \ m \text{ even} \end{cases} \tag{A2}$$

$$F(m,n,p,\ ) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x^m (x^2 - \frac{1}{4})^n}{\zeta^p}\, dx$$

$$= \begin{cases} \left(\dfrac{1}{\eta_o}\right)^p \displaystyle\sum_{k=0}^{\infty} \left(\dfrac{\mathcal{E}}{2\eta_o}\right)^{2k} \dfrac{(p-1+2k)!}{(p-1)!\,(2k)!}\, S(m+2k,n) & , \ m \text{ even} \\[2ex] -\left(\dfrac{1}{\eta_o}\right)^p \displaystyle\sum_{k=0}^{\infty} \left(\dfrac{\mathcal{E}}{2\eta_o}\right)^{2k+1} \dfrac{(p+2k)!}{(p-1)!\,(2k+1)!}\, S(m+2k+1,n) & , \ m \text{ odd} \end{cases} \tag{A3}$$

and

$$P(m,n,\ ) = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^m (x^2 - \tfrac{1}{4})^n \ln\zeta\, dx$$

$$= S(m,n)\ln\eta_o + \mathcal{E}\, S(m+1,n) + \eta_o \sum_{k=2}^{\infty} \frac{(-\mathcal{E}/\eta_o)^k}{k(k-1)}\, S(m+k,n) \tag{A4}$$

Here m and n are zero or positive integers, p is a positive integer, and

$$\eta_o = \frac{1+\eta}{2}, \quad \mathcal{E} = 1-\eta, \quad \zeta = \eta_o + \mathcal{E}x. \tag{A5}$$

The series converge for $0 < \eta < \infty$ and converge for $\eta = 0$ whenever the integrals do. If $\eta = 1$, then $\mathcal{E} = 0$, and the series either terminate after one term or are identically zero. The series converge rapidly for η in the range of interest. For example, to evaluate $F(1,1,1,\frac{1}{2})$ and $F(1,1,1,0.1)$ correct to six significant figures requires 8 and 27 terms respectively.

Let

$$L = DD^* = \frac{d}{dx}\left(\frac{d}{dx} + \frac{\mathcal{E}}{\zeta}\right), \quad L^* = D^*D$$

then

$(u_m,v_n) = S(m+n-2,3)$

$(u_1,v_1) = -1/140$

$(u_1,v_2) = 0$

$(u_1,v_3) = -1/5040$

$(u_2,v_2) = -1/5040$

$(u_2,v_3) = 0$

$(u_3,u_3) = -1/73920$

$$(\zeta v_m, v_n) = \eta_o S(m+n-2,2) + \varepsilon S(m+n-1,2)$$

$$(\zeta v_1, v_1) = \eta_o/30$$

$$(\zeta v_1, v_2) = \varepsilon/840$$

$$(\zeta v_1, v_3) = \eta_o/840$$

$$(\zeta v_1, v_4) = \varepsilon/10080$$

$$(\zeta v_2, v_2) = \eta_o/840$$

$$(\zeta v_2, v_3) = \varepsilon/10080$$

$$(\zeta v_2, v_4) = \eta_o/10080$$

$$(\zeta v_3, v_3) = \eta_o/10080$$

$$(\zeta v_3, v_4) = \varepsilon/88704$$

$$(\zeta v_4, v_4) = \eta_o/88704$$

$$(\zeta u_m, u_n) = \eta_o S(m+n-2,4) + \varepsilon S(m+n-1,4)$$

$$(\zeta u_1, u_1) = \eta_o/630$$

$$(\zeta u_1, u_2) = \varepsilon/27720$$

$$(\zeta u_1, u_3) = \eta_o/27720$$

$$(\zeta u_1, u_4) = \varepsilon/480480$$

$$(\zeta u_2, u_2) = \eta_o/27720$$

$$(\zeta u_2, u_3) = \varepsilon/480480$$

$$(\zeta u_2, u_4) = \eta_o/480480$$

$$(\zeta u_3, u_3) = \eta_o/480480$$

$$(\zeta u_3, u_4) = \varepsilon/5765760$$

$$(\zeta u_4, u_4) = \eta_o/5765760$$

$$(\zeta v_m, u_n) = \eta_o S(m+n-2,3) + \varepsilon S(m+n-1,3)$$

$$(\zeta v_1, u_1) = -\eta_o/140$$

$$(\zeta v_1, u_2) = -\varepsilon/5040$$

$$(\zeta v_1, u_3) = -\eta_o/5040$$

$$(\zeta v_1, u_4) = -\varepsilon/73920$$

$$(\zeta v_2, u_2) = -\eta_o/5040$$

$$(\zeta v_2, u_3) = -\varepsilon/73920$$

$$(\zeta v_2, u_4) = -\eta_o/73920$$

$$(\zeta v_3, u_3) = -\eta_o/73920$$

$$(\zeta v_3, u_4) = \varepsilon/768768$$

$$(\zeta v_4, u_4) = -\eta_o/768768$$

$$(\zeta^2 u_m, v_n) = \eta_o^2 S(m+n-2,3) + 2\eta_o \varepsilon S(m+n-1,3) + \varepsilon^2 S(m+n,3)$$

$$(\zeta^2 u_1, v_1) = -1/140 - \varepsilon^2/5040$$

$$(\zeta^2 u_1, v_2) = -\varepsilon\eta_o/2520$$

$$(\zeta^2 u_1, v_3) = -\eta_o^2/5040 - \varepsilon^2/73920$$

$$(\zeta^2 u_2, v_2) = -\eta_o^2/5040 - \varepsilon^2/73920$$

$$(\zeta^2 u_2, v_3) = -\varepsilon\eta_o/36960$$

$$(\zeta^2 u_3, v_3) = -\eta_o^2/73920 - \varepsilon^2/768768$$

$$(\zeta^{-1} v_m, u_n) = F(m+n-2,3,1,\eta)$$

$$(\zeta^{-2} u_m, v_n) = F(m+n-2,3,2,\eta)$$

$$(\zeta u_m, v_n \ln \zeta) = P(m+n-2,3,\eta)$$

$(\xi v_1, Lv_1) = -\ \eta_o/3 \quad - \varepsilon^2 F(0,2,1,\eta)$

$(\xi v_1, Lv_2) = -\ \varepsilon/30 \quad - \varepsilon^2 F(1,2,1,\eta)$

$(\xi v_1, Lv_3) = -\ \eta_o/60 \quad - \varepsilon^2 F(2,2,1,\eta)$

$(\xi v_1, Lv_4) = -\ \varepsilon/280 \quad - \varepsilon^2 F(3,2,1,\eta)$

$(\xi v_2, Lv_2) = -\ \eta_o/20 \quad - \varepsilon^2 F(2,2,1,\eta)$

$(\xi v_2, Lv_3) = -\ \varepsilon/84 \quad - \varepsilon^2 F(3,2,1,\eta)$

$(\xi v_2, Lv_4) = -\ 3\,\eta_o/560 \quad - \varepsilon^2 F(4,2,1,\eta)$

$(\xi v_3, Lv_3) = -\ 11\,\eta_o/1680 \quad - \varepsilon^2 F(4,2,1,\eta)$

$(\xi v_3, Lv_4) = -\ 11\,\varepsilon/10080 \quad - \varepsilon^2 F(5,2,1,\eta)$

$(\xi v_4, Lv_4) = -\ 23\,\eta_o/20160 \quad - \varepsilon^2 F(6,2,1,\eta)$

$(\xi u_1, Lu_1) = -\ 2\,\eta_o/105 \quad - \varepsilon^2 F(0,4,1,\eta)$

$(\xi u_1, Lu_2) = -\ \varepsilon/1260 \quad - \varepsilon^2 F(1,4,1,\eta)$

$(\xi u_1, Lu_3) = \quad 0 \quad - \varepsilon^2 F(2,4,1,\eta)$

$(\xi u_1, Lu_4) = -\ \varepsilon/55440 \quad - \varepsilon^2 F(3,4,1,\eta)$

$(\xi u_2, Lu_2) = -\ \eta_o/630 \quad - \varepsilon^2 F(2,4,1,\eta)$

$(\xi u_2, Lu_3) = -\ \varepsilon/11088 \quad - \varepsilon^2 F(3,4,1,\eta)$

$(\xi u_2, Lu_4) = -\ \eta_o/13860 \quad - \varepsilon^2 F(4,4,1,\eta)$

$(\xi u_3, Lu_3) = -\ \eta_o/9240 \quad - \varepsilon^2 F(4,4,1,\eta)$

$(\xi u_3, Lu_4) = -\ 31\,\varepsilon/2882880 \quad - \varepsilon^2 F(5,4,1,\eta)$

$(\xi u_4, Lu_4) = -\ 17\,\eta_o/144144 0 \quad - \varepsilon^2 F(6,4,1,\eta)$

$(\xi v_1, L^* v_1) = -\ \eta_o/3$

$(\xi v_1, L^* v_2) = -\ \varepsilon/30$

$(\xi v_1, L^* v_3) = -\ \eta_o/60$

$$(\xi v_2, L^* v_2) = -\ \eta_0/20$$
$$(\xi v_2, L^* v_3) = -\ \mathcal{E}/84$$
$$(\xi v_3, L^* v_3) = -\ 11\,\eta_0/1680$$

$$\begin{aligned}
(\xi u_1, L^2 u_1) &= 4\eta_0/5 &&- 3\mathcal{E}^2\left[12F(2,2,1,\eta) - F(0,2,1,\eta)\right] \\
(\xi u_1, L^2 u_2) &= 3\mathcal{E}/35 &&- 3\mathcal{E}^2\left[16F(3,2,1,\eta) - 2F(1,2,1,\eta)\right] \\
(\xi u_1, L^2 u_3) &= \eta_0/35 &&- 3\mathcal{E}^2\left[21F(4,2,1,\eta) - (7/2)F(2,2,1,\eta) + (1/16)F(0,2,1,\eta)\right] \\
(\xi u_1, L^2 u_4) &= \mathcal{E}/140 &&- 3\mathcal{E}^2\left[27F(5,2,1,\eta) - (11/2)F(3,2,1,\eta) + (3/16)F(1,2,1,\eta)\right] \\
(\xi u_2, L^2 u_2) &= \eta_0/7 &&- 3\mathcal{E}^2\left[20F(4,2,1,\eta) - 3F(2,2,1,\eta)\right] \\
(\xi u_2, L^2 u_3) &= \mathcal{E}/60 &&- 3\mathcal{E}^2\left[25F(5,2,1,\eta) - (9/2)F(3,2,1,\eta) + (1/16)F(1,2,1,\eta)\right] \\
(\xi u_2, L^2 u_4) &= \eta_0/84 &&- 3\mathcal{E}^2\left[31F(6,2,1,\eta) - (13/2)F(4,2,1,\eta) + (3/16)F(2,2,1,\eta)\right] \\
(\xi u_3, L^2 u_3) &= 3\eta_0/140 &&- 3\mathcal{E}^2\left[30F(6,2,1,\eta) - 6F(4,2,1,\eta) + (1/8)F(2,2,1,\eta)\right] \\
(\xi u_3, L^2 u_4) &= 59\mathcal{E}/18480 &&- 3\mathcal{E}^2\left[36F(3,4,1,\eta) + 10F(3,3,1,\eta) + (1/2)F(3,2,1,\eta)\right] \\
(\xi u_4, L^2 u_4) &= 13\eta_0/3696 &&- 3\mathcal{E}^2\left[42F(4,4,1,\eta) + 11F(4,3,1,\eta) + (1/2)F(4,2,1,\eta)\right]
\end{aligned}$$

It is possible of course to derive formulas for the general inner products. For example

$$\begin{aligned}
(\xi v_m, L v_n) = {} & \eta_0\left\{ m(m+1)S(m+n-2,1) - (\tfrac{1}{4})(m-1)(m-2)S(m+n-4,1)\right\} \\
& + \mathcal{E}\left\{ (m+1)^2 S(m+n-1,1) - (\tfrac{1}{4})(m-1)^2 S(m+n-3,1)\right\} \\
& - \mathcal{E}^2 F(m+n-2,2,1,\eta)
\end{aligned}$$